

Effective Levi-Civita Dilaton theory from Metric Affine Dilaton Gravity

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Abstract

We show how a Metric Affine theory of Dilaton gravity can be reduced to an effective Riemannian Dilaton gravity model. A simple generalization of the Obukhov-Tucker-Wang theorem to Dilaton gravity is then presented.

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Among the four fundamental interactions, the two feeble are characterised by dimensional coupling constants, $G_F = (300\text{Gev})^{-2}$ and Newton's coupling constant $G_N = (10^{19}\text{Gev})^{-2}$.

It is well known that interactions with dimensional coupling constants present many problems among which there is the renormalizability.

The success of the Weinberg-Salam model has told us that the weak interaction is characterised by a dimensionless coupling constant and the dimensions of G_F are due to the spontaneous symmetry breaking mechanism, so that $G_F \cong \frac{1}{v_W^2}$ where $v_W \cong 300\text{Gev}$ is the vacuum expectation value of the Higgs field.

The weakness of the weak interaction being related to the large vacuum expectation value of the scalar field [1].

It is believed that similar mechanisms may occur for gravity, which is characterised by a dimensionless coupling constant ξ . The weakness of gravity then would be related to the symmetry breaking at very high energies [2-4]. This is obtained starting from a Dilaton theory which presents Weyl scale invariance. The potential $V(\psi)$ which appears in the action is assumed to have its minimum at $\psi = \sigma$, then when $\psi = \sigma$ the Dilaton theory reduces to the Einstein-Hilbert action with gravitational constant $G_N = \frac{1}{8\pi\xi\sigma^2}$.

In this Letter we investigate in the Tucker-Wang approach to non Riemannian gravity the action:

$$S = \int k\psi^2 R \star 1 + \beta(d\psi \wedge \star d\psi) - V(\psi) \star 1 \quad (1)$$

Where R is the scalar curvature associated with the full non Riemannian connection.

In the Tucker-Wang approach to MAG we choose the metric to be orthonormal $g_{ab} = \eta_{ab} = (-1, 1, 1, 1, \dots)$ and we vary with respect to the coframe e^a and the connection ω^a_b considered as independent gauge potentials.

As we will see the non Riemannian contribution to the Einstein-Hilbert term times ψ^2 is equivalent in the field equations to a kinetic term for the Dilaton and if no torsion terms are explicitly introduced in the action, the coupling ξ is not arbitrary contrary to what happens in Ref [5-7] where ξ is a free parameter.

Once β is fixed we obtain an effective ξ'^{-1} given by $\xi^{-1} + 4\frac{n-1}{n-2}$ (see eq. 22). It has to be observed however that since we have the condition $g_{ab} = \eta_{ab}$, in general the Weyl group for the action (1) is not defined in the usual way

but we may still introduce a Weyl rescaling of the form $e^a \rightarrow f e^a$ for the coframe, with f an arbitrary function of the spacetime..

The variation of (1) with respect to ψ gives:

$$-\beta d \star d\psi + k\psi(R \star 1) - V'(\psi) \star 1 = 0 \quad (2)$$

By considering the variation with respect to the connection we get the equation:

$$D \star (e_a \wedge e^b) = -\frac{2}{\psi} [d\psi \wedge \star(e_a \wedge e^b)] = A(\psi) [d\psi \wedge \star(e_a \wedge e^b)] \quad (3)$$

with $A(\psi) = -\frac{2}{\psi}$.

To prove the previous one we have to observe that the Einstein-Hilbert term which appears in the action (1) can be written as:

$$R \star 1 = R^a_b \wedge \star(e_a \wedge e^b) \quad (4)$$

Where R^a_b are the curvature two forms which are defined by $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$. So we get:

$$R \star 1 = (d\omega^a_b + \omega^a_c \wedge \omega^c_b) \wedge \star(e_a \wedge e^b) \quad (5)$$

Then we have to calculate the variation of:

$$\psi^2 (d\omega^a_b + \omega^a_c \wedge \omega^c_b) \wedge \star(e_a \wedge e^b) \quad (6)$$

We have:

$$\begin{aligned} \psi^2 (d\omega^a_b \wedge \star(e_a \wedge e^b)) = \\ d[\psi^2 \omega^a_b \wedge \star(e_a \wedge e^b)] + \omega^a_b \wedge d[\psi^2 \star(e_a \wedge e^b)] \end{aligned} \quad (7)$$

so (mod d):

$$\begin{aligned} d\omega^a_b \wedge \star(e_a \wedge e^b) \psi^2 = \\ \omega^a_b \wedge d[\psi^2 \star(e_a \wedge e^b)] = \\ \omega^a_b \wedge \psi^2 d(\star(e_a \wedge e^b)) + 2\psi(\omega^a_b \wedge d\psi \wedge \star(e_a \wedge e^b)) \end{aligned} \quad (8)$$

Considering the connection variation and using the definition of the covariant exterior derivative D [8] we obtain formula (3).

The full non Riemannian Einstein Hilbert term can be written as:

$$R \star 1 = \overset{o}{R} \star 1 - \hat{\lambda}^a_c \wedge \hat{\lambda}^c_b \wedge \star(e^b \wedge e_a) - d(\hat{\lambda}^a_b \wedge \star(e^b \wedge e_a)) \quad (9)$$

where $\hat{\lambda}^a_b$ is the traceless part of the non Riemannian part of the connection λ^a_b .

By considering the coframe variation we get then the generalized Einstein equations:

$$\begin{aligned} & k\psi^2 \overset{o}{R}^a_b \wedge \star(e_a \wedge e^b \wedge e_c) - 2k\psi[\hat{\lambda}^a_b \wedge d\psi \wedge \star(e^b \wedge e_a \wedge e_c)] \\ & -\beta[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] + k\psi^2[\hat{\lambda}^a_d \wedge \hat{\lambda}^d_b] \wedge \star(e_a \wedge e^b \wedge e_c) \\ & -V(\psi) \star e_c = 0 \end{aligned} \quad (10)$$

The Cartan equation can be written as:

$$D \star (e^a \wedge e_b) = A(\psi)[d\psi \wedge \star(e^a \wedge e_b)] = F^a_b \quad (11)$$

To solve the previous we need the 0-forms f^{ca}_b defined by $F^a_b = f^{ca}_b \star e_c$. We decompose the nonmetricity and torsion as:

$$\begin{aligned} Q_{ab} &= \hat{Q}_{ab} + \frac{1}{n} g_{ab} Q \\ T_a &= \hat{T}_a + \frac{1}{n-1} (e_a \wedge T) \end{aligned} \quad (12)$$

where $Q_{ab} = Dg_{ab}$, $T_a = de_a + \omega_a^b \wedge e_b$, $Q = Q^a_a$, $T = i_a T^a$.

We have the relations:

$$\begin{aligned} \hat{Q}_{bc} &= \frac{1}{n} g_{bc} (f^d_{da} + f^d_{ad}) e^a - (f_{bac} + f_{bca} - f_{abc}) e^a \\ \hat{T}_c &= \frac{1}{n-1} (e_c \wedge e^a) f^d_{ad} - \frac{1}{2} (e^b \wedge e^a) (f_{bac} + f_{bca} + f_{cab}) \\ T - \frac{n-1}{2n} Q &= \frac{1}{n(n-2)} (f^c_{ac} + (1-n)f^c_{ca}) e^a \end{aligned} \quad (13)$$

We get:

$$f_{cab} = A(\psi)i_c(\star(d\psi \wedge \star(e_a \wedge e_b))) \quad (14)$$

from which we get:

$$\begin{aligned} \hat{Q}^{ab} &= 0 \\ \hat{T}_c &= 0 \end{aligned} \quad (15)$$

and

$$T = \frac{n-1}{2n}Q + \frac{1-n}{n-2}A(\psi)d\psi \quad (16)$$

the solution for the nonmetricity and torsion can then be written as:

$$\begin{aligned} Q_{ab} &= \frac{1}{n}g_{ab}Q \\ T^a &= \frac{1}{2n}(e^a \wedge Q) - \frac{1}{n-2}(e^a \wedge d\psi)A(\psi) \end{aligned} \quad (17)$$

Using the expression of λ^a_b as a function of T^a and Q_{ab} :

$$2\lambda_{ab} = i_a T_b - i_b T_a - (i_a i_b T_c + i_b Q_{ac} - i_a Q_{bc})e^c - Q_{ab} \quad (18)$$

we get:

$$\lambda_{ab} = -\frac{1}{2n}g_{ab}Q + \frac{1}{n-2}A(\psi)(i_a(d\psi)e_b - i_b(d\psi)e_a) \quad (19)$$

and the traceless part:

$$\hat{\lambda}_{ab} = \frac{1}{n-2}A(\psi)(i_a(d\psi)e_b - i_b(d\psi)e_a) \quad (20)$$

By using the previous expression in the generalised Einstein equations we get after some calculations:

$$k\psi^2 \overset{o}{G}_c - \beta'[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c = 0 \quad (21)$$

where $\overset{o}{G}_c = \overset{o}{R}_b^a \wedge \star(e_a \wedge e^b \wedge e_c)$ and:

$$\beta' = \beta + 4k \frac{n-1}{n-2} \quad (22)$$

Then if we choose:

$$\beta = -4k \frac{n-1}{n-2} \quad (23)$$

we get the generalized Einstein equations reduced to:

$$k\psi^2 \overset{\circ}{G}_c - V(\psi) \star e_c = 0 \quad (24)$$

which are equivalent to:

$$k\overset{\circ}{G}_c - V(\psi)\psi^{-2} \star e_c = 0 \quad (25)$$

This are the Einstein equations we would have obtained from an Einstein theory with the potential $V(\psi)\psi^{-2}$.

The non Riemannian contribution to the Einstein-Hilbert term is:

$$\Delta R \star 1 = -\frac{4}{\psi^2} \frac{n-1}{n-2} (d\psi \wedge \star d\psi) \quad (26)$$

so, the equation for ψ becomes:

$$-\beta d \star d\psi + k\psi(\overset{\circ}{R} \star 1) - k\frac{n-1}{n-2} \frac{4}{\psi} (d\psi \wedge \star d\psi) - V'(\psi) \star 1 = 0 \quad (27)$$

Observe the following interesting case.

Suppose we start from the $\beta = 0$ in the action (1) that is:

$$S = \int k\psi^2 R \star 1 - V(\psi) \star 1 \quad (28)$$

then we get the equations:

$$\begin{aligned} k\overset{\circ}{G}_c\psi^2 - 4k\frac{n-1}{n-2} [d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c &= 0 \\ +k\psi \overset{\circ}{R} \star 1 - \frac{4}{\psi} \frac{n-1}{n-2} (d\psi \wedge \star d\psi) - V'(\psi) \star 1 &= 0 \end{aligned} \quad (29)$$

For $n = 4$ we get:

$$\begin{aligned} k\overset{\circ}{G}_c\psi^2 - 6k[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c &= 0 \\ +k\psi \overset{\circ}{R} \star 1 - \frac{6}{\psi} (d\psi \wedge \star d\psi) - V'(\psi) \star 1 &= 0 \end{aligned} \quad (30)$$

The Einstein equations in this case coincide formally with the conformally invariant Einstein equations obtained starting from the action:

$$S = \int k\psi^2 \overset{\circ}{R} \star 1 + 6k(d\psi \wedge \star d\psi) - V(\psi) \star 1 \quad (31)$$

We have to remember however that this equivalence holds with the amendment that the Weyl rescaling is defined for the coframe and not for the metric since g_{ab} is fixed to be orthonormal.

What found above can be extended to more general actions like for example:

$$S = \int k\psi^2 R \star 1 + \beta(d\psi \wedge \star d\psi) + f_1(\psi) \frac{\alpha}{2}(dQ \wedge \star dQ) + f_2(\psi) \frac{\gamma}{2}(Q \wedge \star Q) - V(\psi) \star 1 \quad (32)$$

where Q is the trace of the nonmetricity 1-forms $Q = g_{ab}Q^{ab}$ and $f_1(\psi)$, $f_2(\psi)$ are two arbitrary functions of the Dilaton field ψ .

We get in this case the generalized Einstein equations:

$$k\psi^2 \overset{\circ}{G}_c - \beta'[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c + (33) \\ - f_1(\psi) \frac{\alpha}{2}[dQ \wedge i_c \star dQ + i_c dQ \wedge \star dQ] + f_2(\psi) \frac{\gamma}{2}[Q \wedge i_c \star Q - i_c Q \wedge \star Q] = 0$$

we get the equation for Q .

$$\alpha d(f_1(\psi) \star dQ) + f_2(\psi) \gamma \star Q = 0 \quad (34)$$

Eq. (33) can be considered as a generalization of the Obukhov-Tucker-Wang theorem [9-11] to the Dilaton Gravity action (32).

The equation for ψ (27) contains other two terms which are:

$$\frac{\alpha}{2} f_1'(\psi)(dQ \wedge \star dQ) + \frac{\gamma}{2} f_2'(\psi)(Q \wedge \star Q) \quad (35)$$

If $\beta = -4k \frac{n-1}{n-2}$ then the Einstein equations would reduce to:

$$k\psi^2 \overset{\circ}{G}_c - \quad (36) \\ V(\psi) \star e_c - f_1(\psi) \frac{\alpha}{2}[dQ \wedge i_c \star dQ + i_c dQ \wedge \star dQ] \\ + f_2(\psi) \frac{\gamma}{2}[Q \wedge i_c \star Q - i_c Q \wedge \star Q] = 0$$

The inclusion of torsion terms in the action like $T \wedge \star T$ and $T^c \wedge \star T_c$ would

complicate the analysis since in that case the traceless part of the Cartan equation and then the expression for λ^a_b would be modified [10], the study

of these more general cases as well as the study of further possible generalizations of Obukhov-Tucker-Wang theorem will be considered in following investigations.

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